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Optimum hair strand diameter for minimum free-convection heat transfer from a surface covered with hair

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INTRODUCTION

THE OBJECTIVE of this note is to present an interesting trade-off that may determine the natural size (diameter) of the hair strand in the protective coat of an animal. The general view is that hairs provide an insulation effect by trapping a layer of air, which has a low thermal conductivity. This view, however, is incomplete. An additional function of the hair strand—an unwanted effect in the thermal insulation sense—is to act as a pin fin, and to augment the heat transfer from the otherwise bare area (the skin). Worth noting is that the thermal conductivity of hair material (roughly equal to that of skin, 0.4 W m^{-1} K^{-1}) is almost two orders of magnitude greater than that of air.

It seems that the thermal insulation function served by the hair growth is considerably more interesting than originally thought. Numerous thin hairs may be useful in slowing down the air flow that sweeps the skin, but they can also act as fins to the point that they can defeat the thermal insulation effect associated with the slower air flow.

In geometric terms, the conflict identified above is the mechanism behind the existence of an optimum hair strand diameter that minimizes the overall heat transfer rate from a finite-size portion of skin covered with air. This optimum reveals itself in a surprisingly simple analytical form, if we consider the model outlined in Fig. 1.

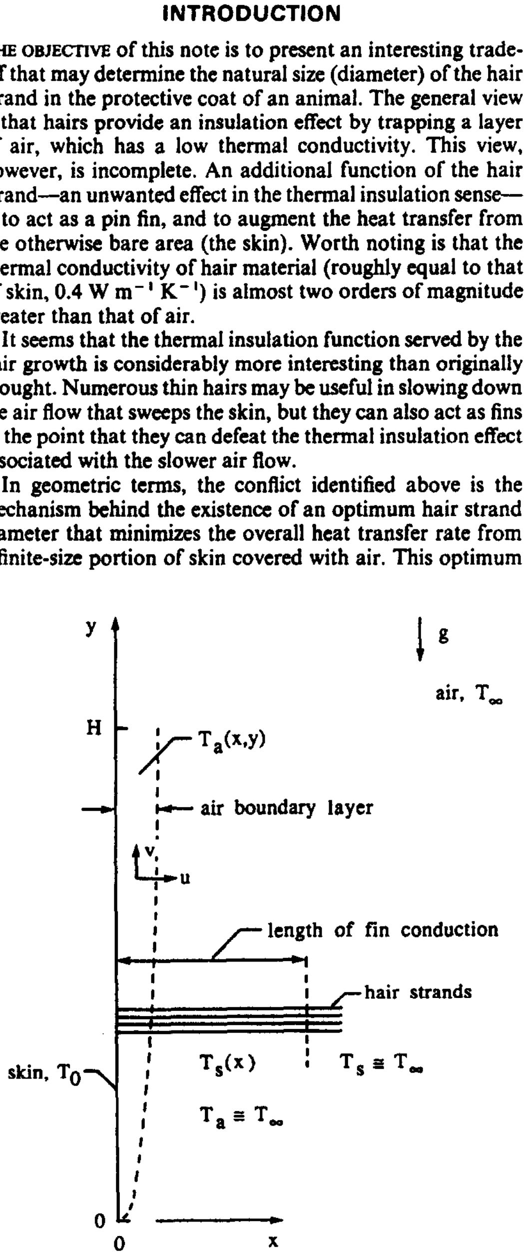

FIG. 1. Vertical skin area, air boundary layer and hair strands that act as fins.

VERTICAL SURFACE

The two-dimensional vertical skin area of height H is covered by a large number of perpendicular hair strands of uniform density n (strands m^{-2}). The hair strands constitute the solid matrix of a porous medium saturated with air. The porosity of this medium is $\phi = 1 - nA_s$, where A_s is the cross-sectional area of the hair strand.

Let $T_a(x, y)$ be the local air temperature averaged over an infinitesimal volume in the air space formed between the hair strands. If u and v are the local air velocity components averaged over the space occupied by air only, then the boundary layer-simplified equation for energy conservation in the air flow is

$$\rho c_P \left(u \frac{\partial T_a}{\partial x} + v \frac{\partial T_a}{\partial y} \right) = k_a \frac{\partial^2 T_a}{\partial x^2} + nhp(T_s - T_a). \quad (1)$$

On the right-hand side, $T_s(x, y)$ is the local temperature of the nearest hair strand, h the strand–air heat transfer coefficient, and p the perimeter of the strand cross-section.

The group $nhp(T_s - T_a)$ represents the volumetric heat source effect that is due to the local contact (or lack of thermal equilibrium) between the air flow and the hair structure. This effect is most noticeable outside the air thermal boundary layer that coats the skin, because in the outer region the temperature of the high-conductivity strand ($T_s \cong T_0$) differs substantially from the air temperature ($T_a \cong T_\infty$). Closer to the skin, the temperature difference $T_s - T_a$ decreases to zero, as both T_s and T_a approach the skin temperature T_0. In the air boundary layer then, we neglect the last term of equation (1), and retain

$$u \frac{\partial T_a}{\partial x} + v \frac{\partial T_a}{\partial y} = \alpha_a \frac{\partial^2 T_a}{\partial x^2}. \quad (2)$$

If the flow obeys Darcy's law, then the momentum equation for air flow in the same boundary layer reduces to (e.g. p. 361 of ref. [1])

$$\frac{\partial v}{\partial x} = \frac{Kg\beta}{\nu\phi} \frac{\partial T_a}{\partial x}. \quad (3)$$

The porosity ϕ appears in the denominator on the right-hand side because this time v is the air velocity averaged only over the space occupied by air. In other words, the product ϕv would be the equivalent of the volume-averaged velocity used in the traditional homogeneous porous medium model.

Along with the mass conservation equation $\partial u/\partial x + \partial v/\partial y = 0$, equations (2) and (3) reproduce the system of equations that has been solved already for boundary layer natural convection along a vertical wall embedded in a homogeneous porous medium [2]. The chief result of the Cheng–Minkowycz solution—the height-averaged heat flux

NOMENCLATURE

A_s	strand cross-sectional area [m^2]
c_P	air specific heat at constant pressure [J kg^{-1} K^{-1}]
D	strand diameter [m]
D_0	diameter of cylinder or sphere [m]
$f_{1,2}$	dimensionless factors, equations (9) and (10)
g	gravitational acceleration [m s^{-2}]
h	heat transfer coefficient [W m^{-2} K^{-1}]
H	height [m]
k	thermal conductivity [W m^{-1} K^{-1}]
k_z	dimensionless constant
K	permeability [m^2]
n	number of hair strands per unit area [m^{-2}]
Nu	overall Nusselt number, equations (8) and (21)
p	perimeter of strand cross-section [m]
q	heat transfer rate [W]
q_1	heat current through the root of one strand [W]
q'	heat transfer rate per unit length [W m^{-1}]
q''	heat flux [W m^{-2}]
Ra	Rayleigh number, equation (12)
Ra_0	Rayleigh number, equation (17)
T	temperature [K]
T_0	skin temperature [K]
T_∞	ambient air temperature [K]
ΔT	temperature difference, $T_0 - T_\infty$ [K]
u, v	air-space averaged velocity components [m s^{-1}]
x, y	Cartesian coordinates. Fig. 1.

Greek symbols

α_a	air thermal diffusivity [m^2 s^{-1}]
β	volume expansivity [K^{-1}]
ν	air kinematic viscosity [m^2 s^{-1}]
ρ	air density [kg m^{-3}]
ϕ	porosity.

Subscripts

a	air property
min	minimum
opt	optimum
s	hair strand property.

q''_a—can be used here, after taking account of the slight change in the model and the notation

$$\frac{q''_a}{\Delta T}\frac{H}{k_a} = 0.888\left(\frac{Kg\beta\Delta TH}{\alpha_a \nu \phi}\right)^{1/2}. \tag{4}$$

The heat transfer rate received by the air flow directly from the skin is $q'_a = q''_a H\phi$, where $H\phi$ is the skin–air contact area per unit length normal to the plane of Fig. 1. In conclusion, the air flow contribution to the total heat transfer rate is

$$q'_a = 0.888 k_a \Delta T \phi^{1/2}\left(\frac{Kg\beta\Delta TH}{\alpha_a \nu}\right)^{1/2}. \tag{5}$$

The other contribution to the total heat transfer rate through the area of height H is due to the hair strands. According to the model that led to the simplified air-energy equation (2), the distance of conduction penetration along the hair strand is considerably larger than the thickness of the air boundary layer. As a fin, the hair strand is bathed by nearly isothermal fluid (T_∞), therefore the heat transfer rate through the root of one hair strand is

$$q_1 = (k_s A_s h p)^{1/2}\Delta T. \tag{6}$$

It is being assumed that the strand is long enough so that the heat transfer through its tip is negligible, and that h is constant. We return to the constant-h assumption in equations (10) and (28). In conclusion, the heat transfer effected by all the hair strands is $q'_s = q_1 nH$, or

$$q'_s = (k_s A_s h p)^{1/2} nH\Delta T. \tag{7}$$

The total heat transfer rate through the surface of height H, per unit area normal to the plane of Fig. 1 is $q' = q'_a + q'_s$. In view of equations (5) and (7), this result can be non-dimensionalized as an overall Nusselt number

$$Nu = \frac{q'}{k_a \Delta T} = 0.888\phi^{1/2}\left(\frac{Kg\beta\Delta TH}{\alpha_a \nu}\right)^{1/2} + \frac{nH}{k_a}(k_s A_s h p)^{1/2}. \tag{8}$$

It is shown later in equations (27) and (28) that in low Reynolds number flow the hair strand diameter D affects both the permeability and the heat transfer coefficient through relations of the type

$$K = D^2 f_1(\phi) \tag{9}$$

$$h = \frac{k_a}{D} f_2(\phi). \tag{10}$$

Substituting also $A_s = \pi D^2/4$ and $p = \pi D$ in equation (8), we obtain

$$Nu = 0.888(\phi f_1 Ra)^{1/2}\frac{D}{H} + 2(1-\phi)\left(\frac{k_s}{k_a} f_2\right)^{1/2}\frac{H}{D} \tag{11}$$

where Ra is the Rayleigh number for free convection in open air

$$Ra = \frac{g\beta\Delta TH^3}{\alpha_a \nu}. \tag{12}$$

Equation (11) shows clearly the trade-off between the air and hair strand heat transfer contributions, as the strand diameter D varies. An optimum strand diameter exists

$$\frac{D_{opt}}{H} = \left(\frac{1-\phi}{0.444}\right)^{1/2}\left(\frac{k_s}{k_a}\frac{f_2}{\phi f_1 Ra}\right)^{1/4} \tag{13}$$

such that the overall heat transfer rate q' (or Nu) reaches a minimum

$$Nu_{min} = 1.776(1-\phi)^{1/2}\left(\phi f_1 f_2 \frac{k_s}{k_a} Ra\right)^{1/4}. \tag{14}$$

Since Ra is proportional to H^3, equation (13) implies that the optimum hair strand diameter is proportional to $H^{1/4}$. In conclusion, the optimum strand diameter for minimum heat transfer is a weak function of the linear size of the animal for which the hair growth serves as insulation.

CYLINDRICAL SURFACE

Analogous conclusions are reached in the case where instead of the vertical plane of Fig. 1, the skin surface has the shape of a long horizontal cylinder. What changes in the analysis is equation (4), that is the formula for the average heat transfer coefficient between the skin and the air boundary layer. According to Cheng [3], the average heat transfer coefficient for a horizontal cylinder of diameter D_0 surrounded by a porous medium with Darcy flow is

$$\frac{q''_a}{\Delta T}\frac{D_0}{k_a} = 0.565\left(\frac{Kg\beta\Delta TD_0}{\alpha_a \nu \phi}\right)^{1/2}. \tag{15}$$

The corresponding direct heat transfer rate to air, per unit length in the axial direction is $q'_a = q''_a \pi D_0 \phi$. The heat transfer rate contribution made by all the hair strands per unit axial length is $q_s' = q_1 n\pi D_0$. The total heat transfer rate $(q' = q_a' + q_s')$ can then be substituted in the overall Nusselt number definition shown on the left-hand side of equation (8). After applying the K and h models, equations (9) and (10), the Nu expression assumes the final form

$$Nu = 1.775(\phi f_1 Ra_0)^{1/2}\frac{D}{D_0} + 2\pi(1-\phi)\left(\frac{k_s}{k_a}f_2\right)^{1/2}\frac{D_0}{D} \tag{16}$$

where Ra_0 is the air-convection Rayleigh number based on cylinder diameter

$$Ra_0 = \frac{g\beta\Delta T D_0^3}{\alpha_a \nu}. \tag{17}$$

The analogy between equations (16) and (11) is evident. The optimum hair strand diameter and the corresponding minimum overall heat transfer rate are

$$\frac{D_{opt}}{D_0} = 1.881(1-\phi)^{1/2}\left(\frac{k_s}{k_a}\frac{f_2}{\phi f_1 Ra_0}\right)^{1/4} \tag{18}$$

$$Nu_{min} = 6.679(1-\phi)^{1/2}\left(\phi f_1 f_2 \frac{k_s}{k_a} Ra_0\right)^{1/4}. \tag{19}$$

Noting that Ra_0 is proportional to D_0^3, we learn that the optimum strand diameter is proportional to $D_0^{1/4}$. This trend is the same as the one revealed by equation (13) for the vertical plane surface covered with hair.

SPHERICAL SURFACE

In the case where the body may be modelled as a sphere of diameter D_0, the average coefficient for heat transfer to the air boundary layer is, cf. Cheng [3]

$$\frac{q_a''}{\Delta T}\frac{D_0}{k_a} = 0.362\left(\frac{Kg\beta\Delta T D_0}{\alpha_a \nu \phi}\right)^{1/2}. \tag{20}$$

Writing $q_a = q_a''\pi D_0^2\phi$ for the total direct heat transfer through the bare portions of the skin, and $q_s = q_1 n\pi D_0^2$ for the contribution made by all the hair strands, we can calculate the total heat transfer rate $q = q_a + q_s$, or the overall Nusselt number

$$Nu = \frac{q}{k_a \Delta T D_0} = 1.137(\phi f_1 Ra_0)^{1/2}\frac{D}{D_0} + 2\pi(1-\phi)\left(\frac{k_s}{k_a}f_2\right)^{1/2}\frac{D_0}{D}. \tag{21}$$

The Rayleigh number Ra_0 has the same definition as in the preceding section, equation (17). The minimization of the Nu expression (21) with respect to the strand diameter D yields

$$\frac{D_{opt}}{D_0} = 2.351(1-\phi)^{1/2}\left(\frac{k_s}{k_a}\frac{f_2}{\phi f_1 Ra_0}\right)^{1/4} \tag{22}$$

$$Nu_{min} = 5.346(1-\phi)^{1/2}\left(\phi f_1 f_2 \frac{k_s}{k_a} Ra_0\right)^{1/4}. \tag{23}$$

The symmetry between these results and equations (13), (14) and (18), (19) is evident. Once again we learn that the hair strand diameter that minimizes the overall heat transfer increases as $D_0^{1/4}$, i.e. as the vertical dimension raised to the power 1/4.

CONCLUSION

The three geometries analyzed in this note—vertical plane, horizontal cylinder and sphere—reinforce the conclusion that the insulation effect of slowly moving air is in competition with the heat-transfer augmentation (finning) effect provided by the hair strands. This competition is visible in the overall heat transfer rate expressions (11), (16), and (21), in which the two terms represent the contributions made by the bare skin and the hair strands, respectively. One implication of this conflict is the existence of an optimum hair strand diameter, for which the overall heat transfer rate is minimum.

The model adopted in this study, Fig. 1, was intentionally simplified so that it would be possible to see the heat-transfer trade-off with the naked eye, as in equations (11), (16), and (21). The validity of this model rests on several limiting assumptions. For example, the hair strand can be treated as a one-dimensional fin only when its Biot number hD/k_s is considerably smaller than 1. In view of equation (10), this condition becomes

$$\frac{k_s}{k_a} > f_2(\phi) \tag{24}$$

in other words, a large k_s/k_a ratio is essential to the validity of this simple model.

Another assumption that was made graphically in Fig. 1 and analytically in the construction of equation (8), is that the distance of conduction penetration along the hair strand is considerably greater than the air boundary layer thickness

$$\left(\frac{k_s A_s}{hp}\right)^{1/2} > H\,Ra^{-1/2}. \tag{25}$$

This condition can be rearranged to read

$$\frac{D}{H} > \left(\frac{4f_2}{Ra}\frac{k_a}{k_s}\right)^{1/2} \tag{26}$$

in which the right-hand side of the inequality is a number considerably smaller than 1. It can be shown that the optimum diameter ratio (13) satisfies inequality (26) when both Ra and k_s/k_a are large numbers.

Less certain is the form of the functions $f_1(\phi)$ and $f_2(\phi)$, that is the effect of porosity on permeability and heat transfer coefficient. Using the friction-factor information listed for tube bundles in cross flow (Fig. 10, pp. 4–100 of Mueller [4]), it can be shown that in the low Reynolds number limit the equivalent 'permeability' of the bundle is represented reasonably well by an expression of the Kozeny type [5]

$$K \cong \frac{\phi^3}{k_z(1-\phi)^2}D^2 \tag{27}$$

in which the empirical constant k_z is consistently of the order of 10^2. Unfortunately, the range of ϕ covered by the tube bundle friction factor data [4] is narrow, $0.42 < \phi < 0.65$, therefore, a special investigation needs to be undertaken to test the validity of the $f_1(\phi)$ form suggested by equation (27).

For the average strand–air friction factor, the analogy between friction and heat transfer suggests that [5] corresponding to equation (27)

$$h \cong \frac{k_z}{4}\frac{1-\phi}{\phi}\frac{k_a}{D}. \tag{28}$$

The same order-of-magnitude conclusion is reached based on the argument that in small-Reynolds-number duct flow h always scales as k_a/D_h (e.g. p. 85 of ref. [1]), where the hydraulic diameter of the air space is $\phi D/(1-\phi)$.

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Coupled heat and mass transfer from a sphere buried in an infinite porous medium

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INTRODUCTION

With the knowledge accumulated from the previous studies on convective heat transfer in porous media, considerable attention has now turned to a more sophisticated problem that takes into account the mass transfer effects. The phenomenon, which is sometimes referred to as 'double-diffusive' or 'thermohaline' convection in geophysical fluid mechanics, has many important applications in energy-related engineering problems, for example, the migration of moisture in fibrous insulation, the spreading of chemical pollutants through water-saturated soil, the cooling of nuclear reactors and the underground disposal of nuclear wastes.

Nield [1] made the first attempt to study the stability of flow in horizontal layers with imposed vertical temperature and concentration gradients for coupled heat and mass transfer by natural convection in a porous medium. Bejan and co-workers [2–5] conducted a series of investigators to study these effects on natural convection for various geometries. In a recent study, Poulikakos [6] extended the results by Bejan [7] to consider buoyancy induced heat and mass transfer from a concentrated source in an infinite porous medium.

The purpose of this study is to analyze another practically important problem of natural convection induced by the combined action of temperature and concentration gradients from a buried sphere. The approach is parallel to that of Poulikakos [6], however, more complicated boundary conditions, i.e. combination of different thermal and concentration boundary conditions, are considered. Emphases have been placed on a fundamental examination of these effects on the flow, temperature and concentration fields.

FORMULATION

Consider a sphere of radius a buried in an infinite porous medium. For heat and mass transfer driven by buoyancy effects, the governing equations based on Darcy's law are simplified by introducing the stream function such that they are given by

$$\frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2\Psi}{\partial R^2}=Ra\left[\left(\cos\theta\frac{\partial\Theta}{\partial\theta}+R\sin\theta\frac{\partial\Theta}{\partial R}\right)-N\left(\cos\theta\frac{\partial C}{\partial\theta}+R\sin\theta\frac{\partial C}{\partial R}\right)\right] \quad (1)$$

$$\frac{1}{R^2\sin\theta}\left(\frac{\partial\Psi}{\partial\theta}\frac{\partial\Theta}{\partial R}-\frac{\partial\Psi}{\partial R}\frac{\partial\Theta}{\partial\theta}\right)=\left[\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\Theta}{\partial R}\right)+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\right] \quad (2)$$

$$\frac{1}{R^2\sin\theta}\left(\frac{\partial\Psi}{\partial\theta}\frac{\partial C}{\partial R}-\frac{\partial\Psi}{\partial R}\frac{\partial C}{\partial\theta}\right)=\frac{1}{Le}\left[\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial C}{\partial R}\right)+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial C}{\partial\theta}\right)\right] \quad (3)$$

where temperature and concentration have been non-dimensionalized as follows:

$$\Theta=\frac{T-T_\infty}{T_0-T_\infty},\ \text{constant temperature}$$
$$\frac{T-T_\infty}{q/ka},\ \text{constant heat flux}$$
$$C=\frac{c-c_\infty}{c_0-c_\infty},\ \text{constant concentration}$$
$$\frac{c-c_\infty}{m/Da},\ \text{constant mass flux.} \quad (4)$$

The subscripts 0 and ∞ denote the condition at the surface of the sphere and at infinity, respectively.

Four different cases are considered in the present study:

(1) a sphere of constant temperature and concentration;
(2) a sphere of constant heat flux and mass flux;
(3) a sphere of constant temperature and mass flux;
(4) a sphere of constant heat flux and concentration.

Therefore, the boundary conditions can be summarized as follows:

at $R=1$,

$\Theta=1$ for constant temperature case

$\frac{\partial\Theta}{\partial R}=-1$ for constant heat flux case

$C=1$ for constant concentration case

$\frac{\partial C}{\partial R}=-1$ for constant mass flux case

$$\frac{1}{R^2\sin\theta}\frac{\partial\Psi}{\partial\theta}=0; \quad (5)$$

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